UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS
General Certificate of Education Advanced Level

FURTHER MATHEMATICS **9231/13**

Paper 1 **October/November 2011**

3 hours

Additional Materials: Answer Booklet/Paper
Graph Paper
List of Formulae (MF10)

8082949944

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of a calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **4** printed pages.

UNIVERSITY *of* CAMBRIDGE
International Examinations

[Turn over

1 Verify that $\dfrac{1}{n^2} - \dfrac{1}{(n+1)^2} = \dfrac{2n+1}{n^2(n+1)^2}$. [1]

Let $S_N = \displaystyle\sum_{r=1}^{N} \frac{2r+1}{r^2(r+1)^2}$. Express S_N in terms of N. [2]

Let $S = \lim\limits_{N\to\infty} S_N$. Find the least value of N such that $S - S_N < 10^{-16}$. [3]

2 Prove by mathematical induction that, for all positive integers n,

$$\frac{d^n}{dx^n}\left(\frac{1}{2x+3}\right) = (-1)^n \frac{n!\,2^n}{(2x+3)^{n+1}}.$$ [6]

3 The equation

$$x^3 + 5x^2 - 3x - 15 = 0$$

has roots α, β, γ. Find the value of $\alpha^2 + \beta^2 + \gamma^2$. [3]

Hence show that the matrix $\begin{pmatrix} 1 & \alpha & \beta \\ \alpha & 1 & \gamma \\ \beta & \gamma & 1 \end{pmatrix}$ is singular. [4]

4 A curve has parametric equations

$$x = 2\sin 2t, \quad y = 3\cos 2t,$$

for $0 < t < \frac{1}{2}\pi$. For the point on the curve where $t = \frac{1}{3}\pi$, find the value of

(i) $\dfrac{dy}{dx}$, [3]

(ii) $\dfrac{d^2y}{dx^2}$. [4]

5 Use de Moivre's theorem to express $\cos^4\theta$ in the form

$$a\cos 4\theta + b\cos 2\theta + c,$$

where a, b, c are constants to be found. [4]

Hence evaluate

$$\int_0^{\frac{1}{4}\pi} \cos^4\theta \, d\theta,$$

leaving your answer in terms of π. [3]

6 Find the general solution of the differential equation

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = \sin 2t. \qquad [6]$$

Describe the behaviour of x as $t \to \infty$, justifying your answer. [2]

7 Show that $\frac{d}{dt}\left(t(1+t^3)^n\right) = (3n+1)(1+t^3)^n - 3n(1+t^3)^{n-1}$. [3]

Let $I_n = \int_0^1 (1+t^3)^n \, dt$. Using the above result, or otherwise, show that

$$(3n+1)I_n = 2^n + 3nI_{n-1}. \qquad [2]$$

Hence evaluate I_3. [4]

8 The curve C has polar equation $r = 1 + \sin\theta$ for $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$. Draw a sketch of C. [2]

The area of the region enclosed by the initial line, the half-line $\theta = \frac{1}{2}\pi$, and the part of C for which θ is positive, is denoted by A_1. The area of the region enclosed by the initial line, and the part of C for which θ is negative, is denoted by A_2. Find the ratio $A_1 : A_2$, giving your answer correct to 1 decimal place. [8]

9 Find a cartesian equation of the plane Π containing the lines

$$\mathbf{r} = 3\mathbf{i} + \mathbf{k} + s(2\mathbf{i} + \mathbf{j} - \mathbf{k}) \quad \text{and} \quad \mathbf{r} = 3\mathbf{i} - 7\mathbf{j} + 10\mathbf{k} + t(\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}). \qquad [4]$$

The line l passes through the point P with position vector $6\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and is parallel to the vector $2\mathbf{i} + \mathbf{j} - 4\mathbf{k}$. Find

(i) the position vector of the point where l meets Π, [3]

(ii) the perpendicular distance from P to Π, [3]

(iii) the acute angle between l and Π. [3]

10 A curve C has equation

$$y = \frac{5(x^2 - x - 2)}{x^2 + 5x + 10}.$$

Find the coordinates of the points of intersection of C with the axes. [2]

Show that, for all real values of x, $-1 \leqslant y \leqslant 15$. [4]

Sketch C, stating the coordinates of any turning points and the equation of the horizontal asymptote. [7]

[Question 11 is printed on the next page.]

11 Answer only **one** of the following two alternatives.

EITHER

The curve C has equation $y = \frac{1}{3}x^{\frac{1}{2}}(3 - x)$, for $0 \leqslant x \leqslant 3$. Find the mean value of y with respect to x over the interval $0 \leqslant x \leqslant 3$. [4]

Show that

$$\frac{\mathrm{d}s}{\mathrm{d}x} = \tfrac{1}{2}\left(x^{-\frac{1}{2}} + x^{\frac{1}{2}}\right),$$

where s denotes arc length, and find the arc length of C. [6]

Find the area of the surface generated when C is rotated through 2π radians about the x-axis. [4]

OR

Find the eigenvalues and corresponding eigenvectors of the matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}.$$ [7]

The linear transformation $\mathrm{T} : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $\mathbf{x} \mapsto \mathbf{A}\mathbf{x}$. Let $\mathbf{e}$, $\mathbf{f}$ be two linearly independent eigenvectors of $\mathbf{A}$, with corresponding eigenvalues λ and μ respectively, and let $\mathit{\Pi}$ be the plane, through the origin, containing $\mathbf{e}$ and $\mathbf{f}$. By considering the parametric equation of $\mathit{\Pi}$, show that all points of $\mathit{\Pi}$ are mapped by T onto points of $\mathit{\Pi}$. [3]

Find cartesian equations of three planes, each with the property that all points of the plane are mapped by T onto points of the same plane. [4]